# TILING POLYGONS BY TRIANGLES OF EQUAL AREAS

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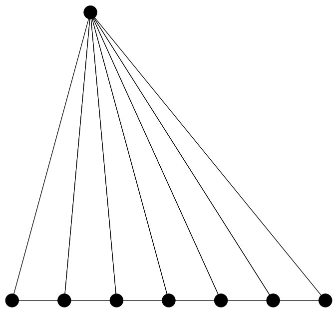
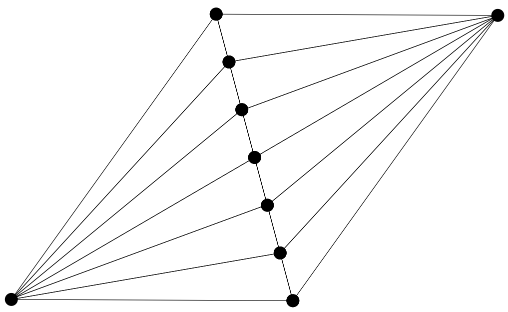
We are considering the problem of covering a plane polygon by a finite number of triangles of equal area, in such a way that the triangles intersect only along their boundaries. Such a covering by $m$ triangles is called an $m$-equidissection. The main theme of this talk will be to establish conditions under which $m$-equidissections exist.

We start our investigation with the brain-dead case: for which $m$ is there an $m$-equidissection of a triangle? The answer is clear: pick an edge of the triangle, cut it into $m$ equal segments, and join each of the division points to the opposing vertex. For $m = 6$, one gets the following picture:



It would be interesting and probably not very hard to figure out how many $m$-equidissections of a triangle there are.

We move on to parallelograms. It is clear that if $m$ is even, we can draw a diagonal in and then use the $m/2$-equidissection of a triangle to get an $m$-dissection; for instance:



What if $m$ is odd? This is not trivial, and we'll encounter some interesting mathematics on our way to answering this question.

## 1. Sperner's lemma

Our arguments will use colorings of the vertices. More precisely, we color vertices using three colors: red, green and blue. We say that a line segment is complete if its two vertices are colored red and green. A polygon is complete if it has an odd number of complete edges. We say that a coloring of a set of points in the plane is good if no line contains points of all three distinct colors.

The first observation is that this notion of completeness of polygons is well-behaved in the following sense. A polygon $P'$ is called a refinement of a polygon $P$ if the vertices of $P'$ consist of the vertices of $P$ plus a finite number of points on the edges of $P$.

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**Lemma.** *Let $P'$ be a refinement of the polygon $P$ and consider a good coloring of the vertices of $P'$. Then $P'$ is complete with respect to this coloring if and only if $P$ is complete with respect to the restriction of this coloring to the vertices of $P$.*

*Proof.* Note that if we take a complete segment and place red and green dots on it, then the number of complete edges is always odd. This is easily seen by induction on $n$, the total number of dots. The case $n = 0$ is clear. Suppose the statement is true for any number of dots $\leq n$. Put $n + 1$ colored dots on a complete edge. There are four cases, according to the colors of the leftmost and rightmost dots:

In the first two cases, we can remove the leftmost segment to get $n$ dots, without changing the number of complete segments. In the last case we can remove the rightmost segment and the number of complete segments will be the same. In the third case, we remove both the leftmost and rightmost segments. Both of them are complete, so the parity of the number of complete segments remains unchanged. Similarly, we see that if we put red and green dots on an edge of type red-red or green-green, the number of complete segments is always even.

What happens when we refine a polygon? We add dots to its edges and we put a good coloring on them. So if we start with a complete edge, we get an odd number of complete edges. If we start with an incomplete edge that has at least one blue vertex, we won't get any complete edges. And if we start with an incomplete edge that's either red-red or green-green, we get an even number of edges. Therefore $P'$ has the same parity of complete edges as $P$. □

We move on to the main combinatorial ingredient of our argument:

**Lemma** (Sperner)**.** *Consider a dissection of a plane polygon $P$, and a red-green-blue coloring of the vertices of the dissection which is such that no line segment contains vertices of all three colors. If $P$ is complete (with respect to this coloring), then there is at least one complete triangle in the dissection.*

*Proof.* Let $T_i$ be the triangles that form the given dissection. A triangle is complete if and only if it has an odd number of complete edges, so

$$\#(\text{complete triangles}) \equiv \sum_i \#(\text{complete edges of } T_i) \pmod 2.$$

Every complete edge of $P$ appears once in this sum, and every complete interior edge appears twice, so we conclude that

$$\#(\text{complete triangles}) \equiv \#(\text{complete edges of } P) \pmod 2.$$

So if $P$ is complete then the number of complete triangles is odd, hence at least 1. □

## 2. Valuations

A valuation on a field $K$ is a function $\phi : K \to \mathbb{R} \cup \infty$ such that

- $\phi(xy) = \phi(x) + \phi(y)$
- $\phi(x + y) \geq \min\{\phi(x), \phi(y)\}$ (triangle inequality)
- $\phi(x) = \infty \iff x = 0$.

A few quick observations: $\phi(1) = \phi(1 \cdot 1)\phi(1) + \phi(1)$ so $\phi(1) = 0$. $0 = \phi(1) = \phi(x(1/x)) = \phi(x) + \phi(1/x)$ so $\phi(1/x) = -\phi(x)$. In particular $\phi(-1) = 0$ and $\phi(-x) = x$ for all $x$. Moreover, if $\phi(x) < \phi(y)$, then the triangle inequality is actually an equality, $\phi(x + y) = \phi(x)$. Why is that? Clearly $\phi(x + y) \geq \phi(x)$. But

$$\phi(x) = \phi(x + y - y) \geq \min\{\phi(x + y), \phi(y)\} \geq \min\{\phi(x), \phi(y)\} = \phi(x).$$

Fix a prime $p$. By unique factorization, any rational number $x$ can be written uniquely as

$$x = p^n \frac{a}{b}, \quad \text{where } p \nmid a, p \nmid b, n \in \mathbb{Z}.$$

We define the $p$-adic valuation by putting $\mathrm{ord}_p(0) = \infty$, $\mathrm{ord}_p(p^n \cdot a/b) = n$. This defines a valuation on $\mathbb{Q}$. By the general theory of valuations $\mathrm{ord}_p$ extends (not uniquely) to a valuation on $\mathbb{R}$. Pick such an extension and denote it also by $\mathrm{ord}_p$. Define the following subsets of the real plane:

$$\begin{aligned} \mathcal{R} &= \{(x, y) \in \mathbb{R}^2 \,|\, \mathrm{ord}_p(x) > 0, \mathrm{ord}_p(y) > 0\} \\ \mathcal{G} &= \{(x, y) \in \mathbb{R}^2 \,|\, \mathrm{ord}_p(x) \leq 0, \mathrm{ord}_p(y) \geq \mathrm{ord}_p(x)\} \\ \mathcal{B} &= \{(x, y) \in \mathbb{R}^2 \,|\, \mathrm{ord}_p(x) > \mathrm{ord}_p(y), \mathrm{ord}_p(y) \leq 0\}. \end{aligned}$$

These three sets partition the plane; we color each point in $\mathcal{R}$ red, each point in $\mathcal{G}$ green and each point in $\mathcal{B}$ blue. Note that translation by any red point preserves colors. Clearly the sum of two red points is red. Take a red point $(x_R, y_R)$ and a green point $(x_G, y_G)$. We have

$$\mathrm{ord}_p(x_R + x_G) = \min\{\mathrm{ord}_p(x_R), \mathrm{ord}_p(x_G)\} = \mathrm{ord}_p(x_G) \leq 0$$

and

$$\mathrm{ord}_p(y_R + y_G) \geq \min\{\mathrm{ord}_p(y_R), \mathrm{ord}_p(y_G)\} \geq \mathrm{ord}_p(x_G) = \mathrm{ord}_p(x_R + x_G).$$

Similarly adding red to blue gives blue. This implies that the coloring is good. Suppose some line $L$ contains all three colors. By picking a red point and translating by its inverse we may assume that $L$ goes through the origin. Now let $(x_G, y_G)$, $(x_B, y_B)$ be a green, resp. a blue point. Then

$$\mathrm{ord}_p(x_G y_B) = \mathrm{ord}_p(x_G) + \mathrm{ord}_p(y_B) < \mathrm{ord}_p(y_G) + \mathrm{ord}_p(x_B) = \mathrm{ord}_p(x_B y_G)$$

contradicting the fact that $x_G y_B = x_B y_G$!

**Lemma.** *If $T$ is a complete triangle with respect to some $p$-adic coloring, then*

$$\mathrm{ord}_p(\mathrm{Area}(T)) \leq -\mathrm{ord}_p(2).$$

*Proof.* Let $T$ have vertices $(x_R, y_R) \in \mathcal{R}$, $(x_G, y_G) \in \mathcal{G}$ and $(x_B, y_B) \in \mathcal{B}$. We have just seen that translating by elements of $\mathcal{R}$ preserves colors, so translating by $(-x_R, -y_R)$ we may assume without loss of generality that $x_R = y_R = 0$. The area of $T$ is the absolute value of

$$\frac{1}{2} \begin{vmatrix} x_G & y_G \\ x_B & y_B \end{vmatrix} = (x_G y_B - x_B y_G)/2.$$

Therefore

$$\mathrm{ord}_p(\mathrm{Area}(T)) = \mathrm{ord}_p(x_G y_B - x_B y_G) - \mathrm{ord}_p(2).$$

By the definitions of $\mathcal{G}$ and $\mathcal{B}$ we have

$$\mathrm{ord}_p(x_B y_G) = \mathrm{ord}_p(x_B) + \mathrm{ord}_p(y_G) > \mathrm{ord}_p(y_B) + \mathrm{ord}_p(x_G) = \mathrm{ord}_p(x_G y_B).$$

The triangle inequality gives

$$\mathrm{ord}_p(x_G y_B - x_B y_G) = \mathrm{ord}_p(x_G y_B) = \mathrm{ord}_p(x_G) + \mathrm{ord}_p(y_B) \leq 0,$$

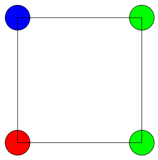
as required. □

Our main result follows quite easily now:

**Theorem 2.1.** *A parallelogram $P$ has no $m$-equidissection with $m$ odd.*

*Proof.* First note that any parallelogram can be mapped into the unit square by an invertible linear transformation, and that such a transformation preserves the ratio of areas. We have therefore reduced the problem to the case where $P$ is the unit square $[0, 1] \times [0, 1]$.

With respect to any $p$-adic coloring of the plane, this square is complete:



Suppose there is an $m$-equidissection of $P$ with $m$ odd. Each triangle has area $1/m$. By Sperner's lemma, there is at least one complete triangle $T$. Its area satisfies

$$-\operatorname{ord}_p(2) \geq \operatorname{ord}_p(\operatorname{Area}(T)) = \operatorname{ord}_p(1/m) = -\operatorname{ord}_p(m).$$

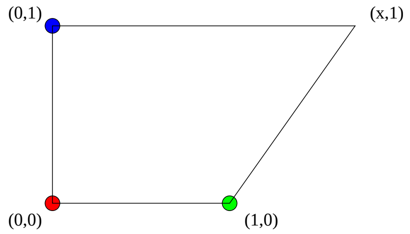
But this holds for all primes $p$, in particular when $p = 2$ we get $-1 \geq 0$, contradiction. $\square$

## 3. Trapezoids

If $P$ is a polygon, we define its spectrum $S(P)$ to be the set of positive integers $m$ such that $P$ can be $m$-equidissected. We saw in the previous section that the spectrum is invariant under affine transformations (compositions of translations and invertible linear transformations), and that the spectrum of a parallelogram is $2\mathbb{N}$, the set of positive even integers. It is clear that if $m \in S(P)$, then $m\mathbb{N} \subset S(P)$. If $S(P)$ consists of all multiples of a given integer, we say that $P$ is principal.

We turn now to the case of a general plane quadrilateral. By applying an affine transformation we may assume that three of the vertices are $(0,0)$, $(0,1)$ and $(1,0)$, and that the fourth vertex is $(x,y)$ with $x \geq y$ (WHY?). We denote this by $Q(x,y)$.

The simplest case is that of a trapezoid, $Q(x,1)$. Note that for any $p$-adic coloring of the plane, $(0,0) \in \mathcal{R}$, $(0,1) \in \mathcal{B}$ and $(1,0) \in \mathcal{G}$. Also $(x,1) \notin \mathcal{R}$ no matter what $x$ is. So $Q(x,1)$ is complete with respect to any $p$-adic coloring:




Surprisingly, this is already complicated enough and we'll encounter a few open problems. But let's start with some of the things that are known.

**Proposition 3.1.**

(a) *If $x$ is transcendental then $S(Q(x,1)) = \emptyset$.*
(b) *If $(r,s) = 1$ and $r/s > 0$ then $S(Q(r/s,1)) = (r+s)\mathbb{N}$.*
(c) *Let $a$, $b$, $c$ be positive integers such that $b^2 - 4ac > 0$ and is not a perfect square. If $x$ satisfies $ax^2 - bx + c = 0$, then the trapezoid $Q(x,1)$ admits an equidissection with $a+b+c$ triangles.*

*Proof.*

(a) Suppose $Q(x,1)$ has an $m$-equidissection, so each triangle has area $(1+x)/2m$. Let $p$ be a prime not dividing $m$. By Sperner's lemma there is a complete triangle, and its area satisfies

$$-\operatorname{ord}_p(2) \geq \operatorname{ord}_p(\operatorname{Area}(T)) = \operatorname{ord}_p(1+x) - \operatorname{ord}_p(2) - \operatorname{ord}_p(m) \quad \Rightarrow \quad \operatorname{ord}_p(x+1) \leq \operatorname{ord}_p(m) = 0.$$

But $x+1$ is transcendental so we can choose any value for $\operatorname{ord}_p(x+1)$, contradiction.

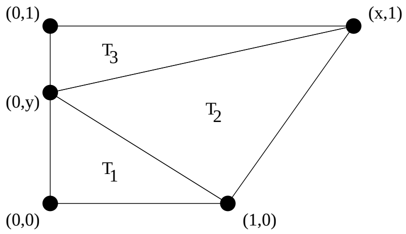
(b) First note that a diagonal of $Q(r/s,1)$ cuts the trapezoid into two triangles whose areas are in the ratio $r : s$. So we can combine an $r$-equidissection of the first triangle with an $s$-equidissection of the second one and get an $r+s$-equidissection of $Q(r/s,1)$. Therefore $(r+s)\mathbb{N}$ is contained in the spectrum.

Now consider an $m$-equidissection of $Q(r/s,1)$ and pick a prime $p$ dividing $r+s$. Since $(r,s) = 1$, we easily see that $\operatorname{ord}_p(r) = \operatorname{ord}_p(s) = 0$. As before, there must be a complete triangle $T$, and its area satisfies

$$-\operatorname{ord}_p(2) \geq \operatorname{ord}_p(\operatorname{Area}(T)) = \operatorname{ord}_p((1+r/s)/2m) = \operatorname{ord}_p(r+s) - \operatorname{ord}_p(s) - \operatorname{ord}_p(m) - \operatorname{ord}_p(2).$$

We get that $\operatorname{ord}_p(m) \geq \operatorname{ord}_p(r+s)$. But this holds for all $p$ dividing $r+s$, so we must have that $(r+s)|m$.

(c) Let $y = ax/(ax+c)$. Since $x > 0$, $a > 0$, $c > 0$ we have $0 < y < 1$. Consider the following dissection of $Q(x,1)$:




The three triangles have the following area ratios:

$$\frac{\operatorname{Area}(T_1)}{\operatorname{Area}(T_3)} = \frac{y}{x(1-y)} = \frac{a}{c}, \quad \frac{\operatorname{Area}(T_2)}{\operatorname{Area}(T_3)} = \frac{1+xy-y}{x-xy} = \frac{ax^2+c}{cx} = \frac{(ax^2-bx+c)+bx}{cx} = \frac{b}{c}.$$

So $\operatorname{Area}(T_1) : \operatorname{Area}(T_2) : \operatorname{Area}(T_3) = a : b : c$, and since $a$, $b$ and $c$ are positive integers we can combine an $a$-equidissection of $T_1$ with a $b$-equidissection of $T_2$ and a $c$-equidissection of $T_3$ to get an $a+b+c$-equidissection of $Q(x,1)$.

□

Another special case of quadrilateral that we may consider is $Q(x,x)$. This will give us an infinite family of quadrilaterals which are not principal:

**Proposition 3.2.**

(a) *Let $r$ and $s$ be relatively prime positive integers with $r$ odd and greater than $s$. Then the spectrum of the convex quadrilateral $Q(r/2s, r/2s)$ contains $2$ and $r + 2sk$ for every nonnegative integer $k$.*

(b) *If $\operatorname{ord}_2(x) > -1$ then the spectrum of $Q(x,x)$ is $2\mathbb{N}$.*

*Proof.*

(a)

(b)

□

Questions:

(a) Does the trapezoid $Q(\sqrt{2},1)$ have an equidissection?

(b) For which algebraic irrational numbers $x$ does $Q(x,1)$ have an equidissection? (Conjecture: When the real conjugates of $x$ are all positive.)

(c) Which trapezoids are principal?

(d) Does $Q(\sqrt{3}/2, \sqrt{3}/2)$ have an odd equidissection?